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Stability analysis adjacent to neutral solutions of the Taylor–Goldstein equation when Howard's formula breaks down

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The Taylor–Goldstein problem for stability of stratified shear flows of inviscid Boussinesq fluids is treated. Perturbation of a known neutral curve is used to obtain the stability characteristics in the neighbourhood of the curve. In the cases that are studied Howard's technique for perturbing neutral modes breaks down. This is related to the vanishing of a coefficient in the expansion of the dispersion relation near the neutral curve. In that case instability may occur on either side of the neutral curve. Examples are used to illustrate how unexpected behaviour arises, such as instability on both sides of a neutral curve.

1. Introduction

The stability properties of some particular stratified shear flows are investigated. The stability characteristics in the neighbourhood of a neutral mode are obtained from the dispersion relation, written as $\alpha^2-\alpha_s^2 = k_1(c-c_s)+k_2(c-c_s)^2+k_3(c-c_s)^3+\dots$ (Engevik 1973*a*, 1975), where α_s and α are the wavenumbers and c_s and c the wave velocities of respectively the neutral mode and the unstable mode contiguous to the neutral one, and k_1, k_2 and k_3 are constants.

k_1 is the inverse of Howard's (1963) formula, which has often been used to investigate the stability characteristics in the neighbourhood of a neutral mode (cf. Drazin & Howard 1966). However, there exist examples for which Howard's formula is not applicable (Huppert 1973; Engevik 1978). This is, for instance, the case when there exists a neutral mode such that k_1 becomes equal to zero, which occurs in the particular flows that we are considering in this paper. In these flows there will always be instability on one side of a neutral curve where $k_1 = 0$, and this unstable side is found by calculating k_2. However, on the other side of the neutral curve nothing can be said conclusively about the stability/instability from the knowledge of k_2 only, and we have to calculate k_3 as well in order to determine whether there is instability or not here. This is the problem we are faced with in this paper. The calculations have revealed new instability regions for the Garcia flow (cf. Holmboe 1962; Miles 1963; Drazin & Howard 1966, p. 77), which is one of the examples studied in this paper.

In §2 the general method for finding the dispersion relation is presented, and it is applied to some particular examples in §3.

In §3.1 we consider the simple stratified Couette flow (Høiland & Riis 1968), for which the dispersion relation can also be expressed in terms of two confluent hyper-

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geometric functions. k_3 is found by expressing the dispersion relation in this way, and by the general method given in §2.

In §3.2 a simple stratified shear flow of infinite vertical extent is discussed. There exist neutral curves along which $k_1 = 0$, corresponding to stationary neutral modes. Application of the general theory in §2 shows that there must be instability on both sides of these curves, and numerical calculations reveal neutral curves corresponding to non-stationary neutral modes.

In §3.3 the Garcia flow is discussed.

2. The general theory

Let the stream function corresponding to an infinitesimal disturbance of a parallel, two-dimensional, inviscid and heterogeneous shear flow be denoted by $\phi(y)\,\mathrm{e}^{\mathrm{i}\alpha(x-ct)}$. If the Boussinesq approximation is made, the amplitude function $\phi(y)$ satisfies the Taylor–Goldstein equation

$$\phi'' + \left\{\frac{JN^2}{(U-c)^2} - \frac{U''}{U-c} - \alpha^2\right\}\phi = 0. \tag{2.1}$$

Here $U(y)$ is the basic-flow velocity, $N(y)$ the buoyancy frequency, J a (representative) Richardson number, α the wavenumber and c the wave velocity (which may be complex). The prime denotes differentiation with respect to y. All variables in (2.1) have been non-dimensionalized with respect to an intrinsic lengthscale L and velocity scale V.

The fluid may be confined between two rigid planes at $y = y_1, y_2$, or may extend to infinity, i.e. y_1 and y_2 may become $-\infty$ and $+\infty$ respectively. $U(y)$ and $N(y)$ are assumed to be analytic functions of $y \in [y_1, y_2]$. The boundary conditions are

$$\left.\begin{array}{ll} \phi = 0 & \text{at } y = y_1, y_2 \\ \text{or} \quad \phi \to 0 & \text{when } y \to \pm\infty. \end{array}\right\} \tag{2.2}$$

It is assumed that there exists a stability boundary, and the amplitude function, the wavenumber, the wave velocity and the Richardson number of the neutral mode on this stability boundary are denoted by ϕ_s, α_s, c_s and J_s respectively. The critical layer corresponding to this neutral mode is at $y = y_s$, where y_s is given by the equation $U(y) = c_s$. We assume that there exists only one critical layer, which lies in the interior of the flow field, and that $U'(y_s) \neq 0$. This means that we do not consider flows with critical layers at the boundaries and the particular problems they pose (see Huppert 1973; Engevik 1978).

Equation (2.1) has a regular singularity at the critical layer y_s. As is well known, this singularity can be removed by introducing dissipative effects. It is found that in the inviscid limit of the linear, diffusive theory there is a $-\pi$ phase shift across the critical layer (Koppel 1964; Baldwin & Roberts 1970; Engevik 1974). We consider the solutions on a contour L that goes around the critical point in accordance with this phase shift across the layer. $\arg(U-c_s)$ is defined to be zero for $U-c_s > 0$ and $-\pi$ for $U-c_s < 0$, and L consists of the line segments $[y_1, y_s-\rho]$ and $[y_s+\rho, y_2]$ on the real axis and the small semicircle C with radius ρ, which lies below or above the critical point according to whether $U'(y_s) > 0$ or $U'(y_s) < 0$.

With the above assumptions, ϕ_s is proportional to either of the two solutions $\phi_\pm = (U-c_s)^{\frac{1}{2}\pm\nu}Y_\pm$, where $\nu = \{\frac{1}{4} - R(y_s)\}^{\frac{1}{2}} \in [0, \frac{1}{2}]$. Here $R(y_s) = J_s N^2(y_s)/(U'(y_s))^2$ is the local Richardson number at the critical layer, Y_+ is analytic on $[y_1, y_2]$, and $Y_\pm(y_s) \neq 0$ (Miles 1961; Engevik 1973*b*). In general $\phi_\pm$ is a many-valued function, and we choose the branch for ϕ_s that is in accordance with the definition of arg $(U-c_s)$ above, i.e. $\phi_\pm = (U-c_s)^{\frac{1}{2}\pm\nu} Y_\pm$ for $U-c_s > 0$ and $\phi_\pm = \exp\{-\mathrm{i}\pi(\frac{1}{2}\pm\nu)\}\,|U-c_s|^{\frac{1}{2}\pm\nu}\, Y_\pm$ for $U-c_s < 0$. $\phi_\pm$ is analytic on L.

We notice that when $\nu = \frac{1}{2}$, which corresponds to $R(y_s) = 0$, both $\phi_+ = (U-c_s)\, Y_+$ and $\phi_- = Y_-$ are analytic on $[y_1, y_2]$ and have no singularity at the critical layer.

Let the amplitude function, the wavenumber, the wave velocity and the Richardson number of a linearly unstable mode contiguous to the neutral one be denoted by ϕ, α, c and J respectively. If $J = J_s$ the dispersion relation for this unstable mode can be written as (Engevik 1973*a*, 1975)

$$\alpha^2 - \alpha_s^2 = \sum_{l=1}^{\infty} k_l (c-c_s)^l, \tag{2.3}$$

where k_l, $l = 1, 2, \ldots$, are constants. Also

$$\phi = \phi_s + \sum_{l=1}^{\infty} \frac{1}{l!} \phi_l (c-c_s)^l, \quad \text{where } \phi_l = \left(\frac{\mathrm{d}^l \phi}{\mathrm{d}c^l}\right)_s. \tag{2.4}$$

The index s means that the expression within the brackets is evaluated at $\alpha = \alpha_s$ and $c = c_s$.

In the following let

$$E = E(y, \alpha, c, J_s) = \frac{J_s N^2}{(U-c)^2} - \frac{U''}{U-c} - \alpha^2,$$

$$E_s = E(y, \alpha_s, c_s, J_s),$$

$$I_l = -\left[\frac{\mathrm{d}^l}{\mathrm{d}c^l}\{(E-E_s)\,\phi\}\right]_s = -\sum_{m=1}^{l} \binom{l}{m} \left[\left(\frac{\partial^m E}{\partial c^m}\right)_s - m!\,k_m\right] \phi_{l-m}, \quad \text{where } \phi_0 = \phi_s.$$

Also let θ_s be a solution of (2.1) with $c = c_s$, $\alpha = \alpha_s$ and $J = J_s$, and such that θ_s and ϕ_s are linearly independent.

It is found (Engevik 1973*a*, 1975) that

$$\phi_l = \phi_s \int_{y_0}^{y} \frac{I_l \theta_s}{W}\,\mathrm{d}t + \theta_s \int_{y}^{y_2} \frac{I_l \phi_s}{W}\,\mathrm{d}t, \tag{2.5}$$

where W is the Wronskian, which is a constant in this case. The integration in (2.5) is along the contour L defined previously, and y_0 can be chosen to be any point on L.

The constant k_l is given by

$$\left.\begin{aligned} k_1 &= \frac{1}{A}\int_L \left(\frac{2J_s N^2}{(U-c_s)^3} - \frac{U''}{(U-c_s)^2}\right)\phi_s^2\,\mathrm{d}y, \quad \text{where } A = \int_L \phi_s^2\,\mathrm{d}y, \\ k_l &= \frac{1}{A}\left[\sum_{m=1}^{l-1} \frac{1}{(l-m)!}\int_L \left\{\frac{1}{m!}\left(\frac{\partial^m E}{\partial c^m}\right)_s - k_m\right\}\phi_{l-m}\,\phi_s\,\mathrm{d}y \right. \\ &\qquad \left. + \frac{1}{l!}\int_L \left(\frac{\partial^l E}{\partial c^l}\right)_s \phi_s^2\,\mathrm{d}y\right], \quad l = 2, 3, \ldots. \end{aligned}\right\} \tag{2.6}$$

We observe that ϕ_l and k_l can be obtained from (2.5) and (2.6) once ϕ_s and θ_s are known. First k_1 can be obtained when ϕ_s is known; then ϕ_1 can be found since k_1 is known, and so on. k_l can be obtained when $\phi_1, \ldots, \phi_{l-1}$ and $k_1, \ldots, k_{l-1}$ are known, and when k_l is also known ϕ_l can be found.

The formulae (2.6) will be used to examine whether there exist unstable modes contiguous to the neutral ones that are found in the examples in §3. In these examples the velocity profile $U(y)$ is an odd function and the buoyancy frequency $N(y)$ is an even function of y. Then we know that if $\phi(y)$ is an eigenvalue function with wave velocity c then $\phi^*(-y)$ is also an eigenvalue function with wave velocity c^*, where the asterisk means complex conjugate. This property can be used to say something about k_l in the cases when the neutral solution is stationary, i.e. $c_s = 0$. It is found that k_l must be purely imaginary when l is odd, and real when l is even. To know this simplifies the calculation of k_l.

In the examples in §3 stationary neutral solutions have been found, located on curves in the (α, J)-plane. Along some of these curves $k_1 \neq 0$, along others $k_1 = 0$. When $k_1 \neq 0$ there will be instability on one side of the curve only. However, when $k_1 = 0$ there may exist unstable modes on both sides of the neutral curve, and we have to calculate k_3 in order to decide whether there is instability on one or both sides. If we put $c_s = 0$, $c = c_r + ic_i$, $k_1 = 0$, $k_2 \neq 0$ and $k_3 = ik_{3i}$ into (2.3) we obtain the following results:

(i) on that side of the neutral curve where $\alpha^2 - \alpha_s^2$ and k_2 have opposite signs there is instability with

$$\left.\begin{aligned} c_i &= \left(-\frac{\alpha^2-\alpha_s^2}{k_2}\right)^{\frac{1}{2}} + O(\alpha^2-\alpha_s^2), \\ c_r &= 0 \end{aligned}\right\} \text{ when } \alpha \to \alpha_s; \tag{2.7}$$

(ii) on the side of the neutral curve where $\alpha^2 - \alpha_s^2$ and k_2 have the same sign there is instability if k_2 and k_{3i} have opposite signs, and we find that

$$\left.\begin{aligned} c_r &= \pm\left(\frac{\alpha^2-\alpha_s^2}{k_2}\right)^{\frac{1}{2}} + O((\alpha^2-\alpha_s^2)^{\frac{3}{2}}), \\ c_i &= -\frac{k_{3i}}{2k_2}\left(\frac{\alpha^2-\alpha_s^2}{k_2}\right) + O((\alpha^2-\alpha_s^2)^2) \end{aligned}\right\} \text{ when } \alpha \to \alpha_s. \tag{2.8}$$

We notice that on one side of the neutral curve $c_r = 0$ ('principle of exchange of stabilities'); on the other side there are two unstable waves with opposite directions of propagation.

3. Examples

3.1. *Stratified Couette flow*

In this first example we consider stratified Couette flow, i.e. $U = y$, $N^2 = y^2$ and $y_2 = -y_1 = 1$ (Høiland & Riis 1968). There exist the stationary neutral modes

$$c_s = 0, \quad \phi_s = \sin n\pi y, \quad \alpha_s^2 = J - (n\pi)^2, \quad n = 1, 2, \ldots, n_1, \tag{3.1}$$

$$c_s = 0, \quad \phi_s = \cos (n-\tfrac{1}{2})\pi, \quad \alpha_s^2 = J - ((n-\tfrac{1}{2})\pi)^2, \quad n = 1, 2, \ldots, n_2 \tag{3.2}$$

where n_1 and n_2 are the largest integers that satisfy $J-(n_1\pi)^2 \geqslant 0$ and $J-((n_2-\frac{1}{2})\pi)^2 \geqslant 0$ respectively.

It has been shown by Engevik (1973*b*) that $k_1 = 0$ for the modes given by (3.1). This is in agreement with the result $(\partial c/\partial \alpha^2)_J = \infty$ of Huppert (1973), who used Howard's (1963) formula, which is the inverse of k_1. Therefore we have to calculate both k_2 and k_3 in order to find the stability characteristics in the neighbourhood of these neutral modes. k_2 has also been found previously by different methods (Huppert 1973; Banks & Drazin 1973; Engevik 1973*a*). By using the general formula in (2.6), k_2 becomes

$$k_2 = 6Jn\pi \,\mathrm{Si}\,(2\pi n) - \frac{2J^2}{n\pi}\,\mathrm{Si}\,(2\pi n)\,\mathrm{Cin}\,(2\pi n)$$
$$-\frac{2J^2}{n\pi}\left[\int_0^1 \frac{1-\cos(2\pi n y)}{y}\,\mathrm{Si}\,(2\pi n y)\,\mathrm{d}y - \int_0^1 \frac{\sin(2\pi n y)}{y}\,\mathrm{Cin}\,(2\pi n y)\,\mathrm{d}y\right], \quad (3.3)$$

where the sine and cosine integrals are defined respectively by

$$\mathrm{Si}\,(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{d}t, \quad \mathrm{Cin}\,(x) = \int_0^x \frac{1-\cos t}{t}\,\mathrm{d}t.$$

By a different method (see below) Huppert (1973) calculated the inverse of k_2, and obtained an expression which has been shown to be equivalent to that of (3.3) (Banks & Drazin 1973; Engevik 1973*a*). Since it turns out that k_2 is negative, it follows from (2.7) that there is instability for $\alpha > \alpha_s$. However, nothing conclusively can be said about the stability characteristics for $\alpha < \alpha_s$ until we have calculated k_3. We applied the formula (2.6) to find k_3, and, taking into account that k_3 must be purely imaginary, the calculation turned out to be quite simple. We obtained

$$k_3 = \mathrm{i}k_{31} = \mathrm{i}\pi\left\{4J(n\pi)^2 - 6J^2\,\mathrm{Cin}\,(2n\pi) + \frac{2J^3}{(n\pi)^2}(\mathrm{Cin}\,(2n\pi))^2\right\}. \quad (3.4)$$

It is easily found that $k_{31} > 0$ along the neutral curves where $k_1 = 0$, and therefore there is also instability for $\alpha < \alpha_s$, which follows from (2.8). This result is in agreement with the numerical calculations of Huppert (1973).

We have chosen to consider this example, because in this case both k_2 and k_3 can be found without using the general theory, and therefore we have the possibility of checking the result (3.4) that is obtained from the general theory. In this simple case the dispersion relation can be expressed in terms of two confluent hypergeometric functions (Huppert 1973). It becomes

$$M(a, b, z(1))\,U(a, b, z(-1)) - M(a, b, z(-1))\,U(a, b, z(1)) = 0 \quad (3.5)$$

where $M(a, b, z)$ and $U(a, b, z)$ are the two linearly independent confluent hypergeometric functions, which are defined in Abramowitz and Stegun (1965), and

$$\left.\begin{aligned} a &= \tfrac{1}{2} + (\tfrac{1}{4} - Jc^2)^{\frac{1}{2}} - Jc(\alpha^2 - J)^{-\frac{1}{2}}, \\ b &= 1 + (1 - 4Jc^2)^{\frac{1}{2}}, \\ z(y) &= 2(\alpha^2 - J)^{\frac{1}{2}}(y - c). \end{aligned}\right\} \quad (3.6)$$

In the calculations we have used the integral representations of $M(a, b, z)$ and $U(a, b, z)$ given respectively by equations (13.2.1) and (13.2.6) in Abramowitz & Stegun.

If we expand the left-hand side of (3.5) about the mode (3.1) we get

$$-\frac{\mathrm{i}(\alpha^2-\alpha_s^2)}{2(n\pi)^3}+\left[\frac{3\mathrm{i}J\,\mathrm{Si}\,(2n\pi)}{(n\pi)^2}-\frac{\mathrm{i}J^2}{(n\pi)^3}\left\{\frac{1}{2}\int_0^1 \cos(2n\pi t)\left(\log\frac{t}{1-t}\right)^2 \mathrm{d}t\right.\right.$$
$$\left.\left.+\frac{1}{n\pi}\,\mathrm{Si}\,(2n\pi)\,\mathrm{Cin}\,(2n\pi)\right\}\right]c^2$$
$$+\left[\frac{2J}{n\pi}\left\{3\,\mathrm{Si}\,(2n\pi)-\pi\right\}+\frac{J^2}{(n\pi)^2}\left\{\frac{2\,\mathrm{Cin}\,(2n\pi)}{n}-\frac{2\,\mathrm{Si}\,(2n\pi)\,\mathrm{Cin}\,(2n\pi)}{n\pi}\right.\right.$$
$$\left.+\int_1^\infty \cos(2n\pi t)\left(\log\frac{t}{t-1}\right)^2 \mathrm{d}t-\frac{1}{2}\int_0^1 \cos(2n\pi t)\left(\log\frac{t}{1-t}\right)^2 \mathrm{d}t\right\}$$
$$\left.-\frac{J^3\pi}{(n\pi)^5}(\mathrm{Cin}\,(2n\pi))^2\right]c^3-\frac{(\alpha^2-\alpha_s^2)\,c}{(n\pi)^2}+\ldots=0. \tag{3.7}$$

From (3.7) it follows that

$$\alpha^2-\alpha_s^2=\left[6Jn\pi\,\mathrm{Si}\,(2n\pi)-J^2\left\{\int_0^1 \cos(2n\pi t)\left(\log\frac{t}{1-t}\right)^2 \mathrm{d}t\right.\right.$$
$$\left.\left.+\frac{2}{n\pi}\,\mathrm{Si}\,(2n\pi)\,\mathrm{Cin}\,(2n\pi)\right\}\right]c^2+O(c^3), \tag{3.8}$$

The coefficient of c^2 has previously been found by Huppert, and is equivalent to the expression (3.3) for k_2.

If we put the expression (3.8) for $\alpha^2-\alpha_s^2$ into the term $(\alpha^2-\alpha_s^2)\,c/(n\pi)^2$ in (3.7) we get

$$\alpha^2-\alpha_s^2=\left[6Jn\pi\,\mathrm{Si}\,(2n\pi)-J^2\left\{\int_0^1 \cos(2n\pi t)\left(\log\frac{t}{1-t}\right)^2 \mathrm{d}t+\frac{2}{n\pi}\,\mathrm{Si}\,(2n\pi)\,\mathrm{Cin}\,(2n\pi)\right\}\right]c^2$$
$$+\mathrm{i}\pi\left[4J(n\pi)^2-6J^2\,\mathrm{Cin}\,(2n\pi)+\frac{2J^3}{(n\pi)^2}(\mathrm{Cin}\,(2n\pi))^2\right]c^3+\ldots, \tag{3.9}$$

where we have used

$$\int_1^\infty \cos(2n\pi t)\left(\log\frac{t}{t-1}\right)^2 \mathrm{d}t=-\frac{1}{2}\int_0^1 \cos(2n\pi t)\left(\log\frac{t}{1-t}\right)^2 \mathrm{d}t+\frac{\mathrm{Cin}\,(2n\pi)}{n},$$

which can be shown by integrating $\mathrm{e}^{\mathrm{i}2n\pi z}(\log z-\log(z-1))^2$ along a proper contour in the complex z-plane.

Equation (3.9) shows that we have found the same expression for k_3 as the one obtained by the general theory. However, k_3 is more easily found by the general theory.

3.2. *Stratified shear flow of infinite vertical extent*

In this example we consider the model

$$U(y)=\begin{cases}1 & (y\in(\frac{1}{2}\pi,\infty)),\\ \sin y & (y\in[-\frac{1}{2}\pi,\frac{1}{2}\pi]),\\ -1 & (y\in(-\infty,-\frac{1}{2}\pi)),\end{cases}$$

$$N^2(y) = \begin{cases} 0 & (y \in (\tfrac{1}{2}\pi, \infty)), \\ \sin^2 y & (y \in [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi]), \\ 0 & (y \in (-\infty, -\tfrac{1}{2}\pi)), \end{cases}$$

$$y_2 = -y_1 = \infty.$$

Let λ be defined by $\lambda^2 = J_s + 1 - \alpha_s^2$. It is found that there exist the following stationary modes:

the even stationary modes are

$$\left.c_s = 0, \quad \phi_s = \begin{cases} \cos(\tfrac{1}{2}\lambda\pi)\exp[-\alpha_s(y-\tfrac{1}{2}\pi)] & (y \in (\tfrac{1}{2}\pi, \infty)), \\ \cos\lambda y & (y \in [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi]), \\ \cos(\tfrac{1}{2}\lambda\pi)\exp[\alpha_s(y+\tfrac{1}{2}\pi)] & (y \in (-\infty, -\tfrac{1}{2}\pi)), \end{cases}\right\} \quad (3.10a, b)$$

where α_s and J_s are connected through

$$\tan\tfrac{1}{2}\lambda\pi = \alpha_s/\lambda; \quad (3.10c)$$

the odd stationary modes are

$$\left.c_s = 0, \quad \phi_s = \begin{cases} \sin(\tfrac{1}{2}\lambda\pi)\exp[-\alpha_s(y-\tfrac{1}{2}\pi)] & (y \in (\tfrac{1}{2}\pi, \infty)), \\ \sin\lambda y & (y \in [-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi]), \\ -\sin(\tfrac{1}{2}\lambda\pi)\exp[\alpha_s(y+\tfrac{1}{2}\pi)], & (y \in (-\infty, -\tfrac{1}{2}\pi)), \end{cases}\right\} \quad (3.11a.\ b)$$

where α_s and J_s are connected through

$$\tan\tfrac{1}{2}\lambda\pi = -\lambda/\alpha_s. \quad (3.11c)$$

The relations between α_s and J_s in (3.10*c*) and (3.11*c*) give the neutral curves along which $c_s = 0$ in figure 1. The neutral curves corresponding to the even modes are denoted by (11), (12) and (13), and the ones corresponding to the odd modes by (21), (22) and (23).

The theory in §2 can be applied in this case too, in spite of the fact that $U(y)$ and $N(y)$ are only piecewise-analytic on $(-\infty, \infty)$. For the theory to be applicable it is sufficient that $U(y)$ and $N(y)$ are analytic in the vicinity of the critical layer, which is the case in this example.

By introducing the modes (3.10) into k_1 given by (2.6) we obtain

$$\left.\begin{aligned} k_1 &= \mathrm{i}\pi(2J_s+1)/A_1, \\ \text{where} \qquad A_1 &= \frac{\cos^2\frac{1}{2}\lambda\pi}{\alpha_s} + \tfrac{1}{2}\pi\left(1 + \frac{\sin\lambda\pi}{\lambda\pi}\right), \end{aligned}\right\} \quad (3.12)$$

which yields instability in the neighbourhood to the left of the curves (11), (12) and (13).

By introducing the modes (3.11) into the formulae in (2.6) we get

$$k_1 = 0, \quad (3.13)$$

$$\left.\begin{aligned} k_2 &= \frac{2}{A_2}\int_0^{\frac{1}{2}\pi} H(y)\,\mathrm{d}y, \\ \text{where} \qquad H(y) &= \frac{(3J_s+1)\sin^2\lambda y}{\sin^2 y} - \frac{(2J_s+1)^2 I_0(y)\sin 2\lambda y}{\lambda\sin y}, \\ I_0(y) &= \int_y^{\frac{1}{2}\pi}\frac{\sin^2\lambda t}{\sin t}\,\mathrm{d}t, \end{aligned}\right\} \quad (3.14)$$

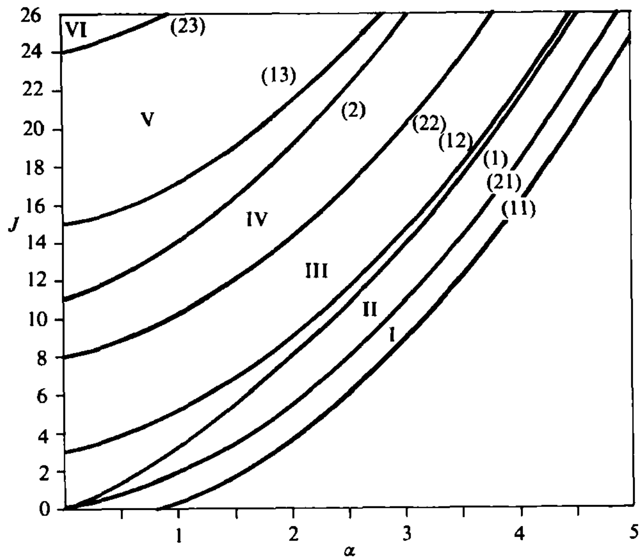

FIGURE 1. The unstable regions are denoted by roman numerals: $c_r = 0$ in I, III and V, and $c_r \neq 0$ in II, IV and VI. The neutral modes corresponding to the neutral curves (1) and (2) are non-stationary, while those corresponding to the other neutral curves are stationary.

$$k_3 = \mathrm{i}\frac{\pi}{A_2}\left\{\frac{(2J_s+1)^3 I_0^2(0)}{\lambda^2} - 2(2J_s+1)(3J_s+1)I_0(0) + \lambda^2(4J_s+1)\right\}, \tag{3.15}$$

where

$$A_2 = \frac{\sin^2\frac{1}{2}\lambda\pi}{\alpha_s} + \tfrac{1}{2}\pi\left(1 - \frac{\sin\lambda\pi}{\lambda\pi}\right).$$

From (3.14) it follows that $k_2 \sim -2\alpha_s^2$ when $\alpha_s \to 0$ along the curve (21), which is the same asymptotic expression for k_2 as was found for the Holmboe and the Garcia flows (Engevik 1982).

Numerical calculations have shown that $k_2 < 0$ along the curves (21), (22) and (23). From (2.7) it follows that there exist unstable modes contiguous to ϕ_s for $\alpha > \alpha_s$. To decide whether there is instability to the left of these curves also, we have to calculate k_3 along the curves. Numerical calculations show that there exists a point $(\alpha_s, J_s) = (\alpha_0, J_0)$ on the curves (21) such that $k_{3i} < 0$ for $\alpha_s < \alpha_0$ $(J_s < J_0)$, and that $k_{3i} > 0$ for $\alpha_s > \alpha_0$ $(J_s > J_0)$. Therefore there is no instability in the neighbourhood to the left of the curve (21) when $\alpha_s < \alpha_0$, but there exist unstable modes to the left of this curve when $\alpha_s > \alpha_0$. This follows from (2.8). Approximate values of α_0 and J_0 are 0.12 and 0.17 respectively.

It is found that $k_{3i} > 0$ along the curves (22) and (23), and therefore there is instability in the neighbourhood of these curves to the left also (see figure 1).

We now know that there exist unstable modes to the left of the neutral curve (21), (22) and (23), and we also know the asymptotic value of c given by (2.8). We therefore carried through a numerical solution of the eigenvalue problem (2.1)–(2.2), taking the asymptotic value given by (2.8) as the starting value of c. The numerical results are shown in figure 1. New neutral curves, denoted by (1) and (2) in figure 1, are found. Only the part of the neutral curve (1) that forms a part of the stability boundary is calculated i.e. the part with α-values greater or equal to α_0.

The neutral modes corresponding to the neutral curves (1) and (2) are not stationary. In figures 2–4 it is shown how c_r and c_i vary with α from curve (21) to

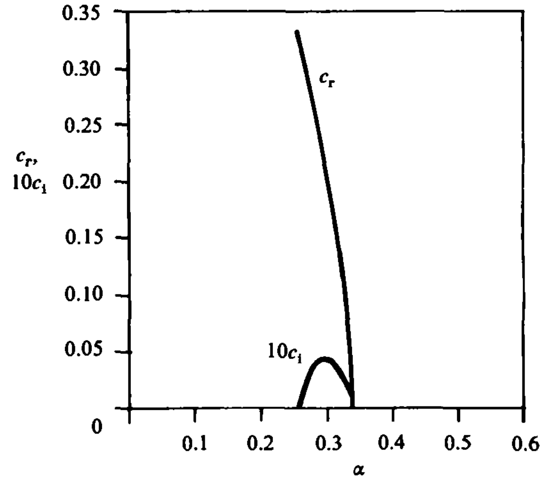

FIGURE 2. The variation of c_r and c_i with α in the unstable region II in figure 1 for $J = 0.5$.

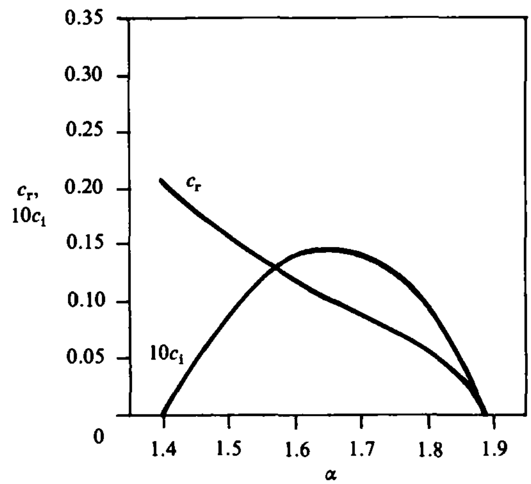

FIGURE 3. The variation of c_r and c_i with α in the unstable region II in figure 1 for $J = 5$.

curve (1) in the unstable region II for some fixed values of J. We see that c_r increases from curve (21), where it is equal to zero, to curve (1). Analogous behaviour of c_r and c_i is found between the curves (22) and (2).

3.3. *The Garcia flow*

In this last example we consider the Garcia flow (cf. Holmboe 1962; Miles 1963, Drazin & Howard 1966, p. 77):

$$U = \tanh y, \quad N^2 = 3\,\mathrm{sech}^2 y \tanh^2 y, \quad y_2 = -y_1 = \infty. \tag{3.16}$$

Garcia found the neutral modes

$$c_s = 0, \quad J_s = \tfrac{1}{3}(\alpha_s - 1)(\alpha_s + 2), \quad \phi_s = (\mathrm{sech}\, y)^{\alpha_s}, \tag{3.17}$$

$$c_s = 0, \quad J_s = \tfrac{1}{3}\alpha_s(\alpha_s + 3), \quad \phi_s = \tanh y\, (\mathrm{sech}\, y)^{\alpha_s}, \tag{3.18}$$

which are located on the two neutral curves denoted respectively by (11) and (21) in figure 5. Miles (1963) found that in this model there exist an infinite number of

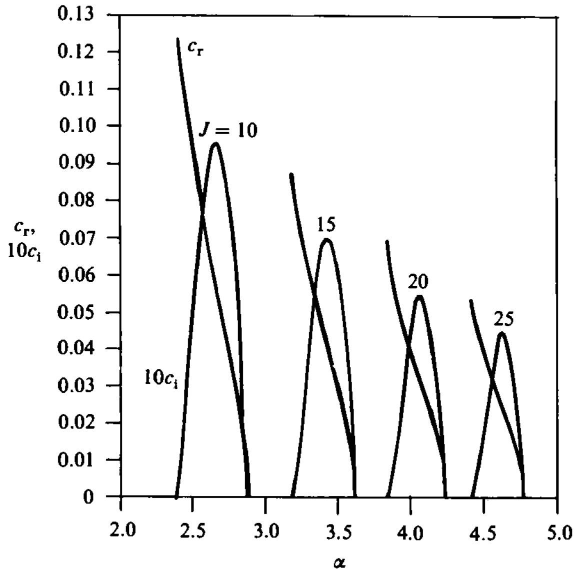

FIGURE 4. The variation of c_r and c_i with α in the unstable region II in figure 1 for $J = 10$, 15, 20 and 25.

neutral curves corresponding to neutral stationary modes. Some of these curves, denoted by (12), (22), (13), (23) and (14), are sketched in figure 5.

It is found (Engevik 1982) that along the curve (11) $k_1 = \mathrm{i}k_{1\mathrm{i}}$ is purely imaginary and that $k_{1\mathrm{i}} > 0$, which yields instability for $\alpha < \alpha_s$. Further, it is found (Engevik 1982) that along the curve (21)

$$\left.\begin{aligned} k_1 &= 0, \\ k_2 &= -\frac{2\alpha_s^2(2\alpha_s^3+9\alpha_s^2+10\alpha_s+1)}{\alpha_s+1}. \end{aligned}\right\} \tag{3.19}$$

We see from (3.19) that $k_2 < 0 \,\forall\, \alpha_s$, which yields instability for $\alpha > \alpha_s$. In order to find whether there are unstable modes in the neighbourhood to the left of the curve (21) we had to calculate k_3. We obtained

$$k_3 = \frac{2\mathrm{i}\pi\alpha_s^2(\alpha_s^4+6\alpha_s^3+11\alpha_s^2+5\alpha_s-2)}{(\alpha_s+1)^2 B(\frac{3}{2},\alpha_s)}, \tag{3.20}$$

where $B(r, s)$ denotes the beta function. It follows from (3.20) that there exists a point $(\alpha_s, J_s) = (\alpha_0, J_0)$ on the curve (21) in figure 5 such that

$$\left.\begin{aligned} &k_{3\mathrm{i}} > 0 \quad \text{for } \alpha_s > \alpha_0 \approx 0.25 \quad (J_s > J_0 = \tfrac{1}{3}\alpha_0(\alpha_0+3) \approx 0.27), \\ &k_{3\mathrm{i}} < 0 \quad \text{for } \alpha_s < \alpha_0 \quad (J_s < J_0), \end{aligned}\right\} \tag{3.21}$$

which shows that there are neighbouring unstable modes also to the left of the curve (21) for $\alpha_s > \alpha_0$, but that there are no such modes for $\alpha_s < \alpha_0$.

Numerical calculations have revealed new instability regions for the Garcia flow and new neutral curves denoted by (1), (2) and (3) (see figure 5). Only the part of the curve (1) that is a boundary between a stable and an unstable region has been calculated. The neutral modes corresponding to these new neutral curves are not

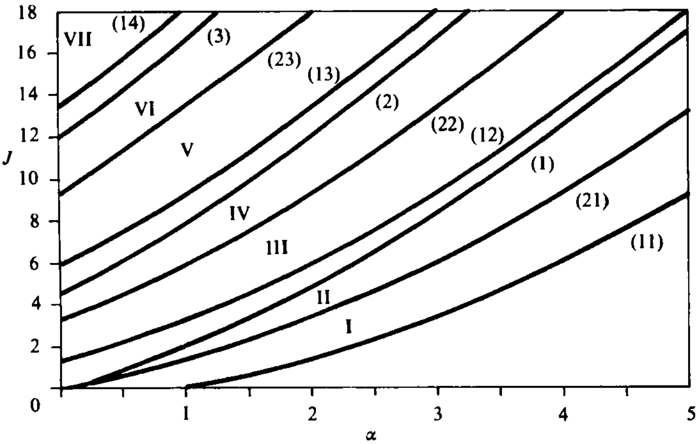

FIGURE 5. The unstable regions are denoted by roman numerals: $c_r = 0$ in I, III, V, VII, and $c_r \neq 0$ in II, IV, VI. The neutral modes corresponding to the neutral curves (1), (2) and (3) are non-stationary, while those corresponding to the other neutral curves are stationary.

stationary. The variations of c_r and c_i through the unstable region II, IV and VI are found to be analogous to those shown in figures 2–4.

The result (3.20) is similar to (3.15). In the region close to the origin in the (α, J)-plane it is the curve (21), along which $c_s = 0$, that represents the boundary between the stable and the unstable regions in both of the examples in §§3.2 and 3.3. This is to be expected from the results of Drazin & Howard (1961, 1963) concerning the stability characteristics for a profile of shear-layer type in the limit $\alpha \to 0$ with J/α fixed (cf. Drazin & Howard 1966). It is therefore to be expected that the neutral curve (1) in figures 1 and 5 should not be a stability boundary close to the origin in the (α, J)-plane.

4. Conclusion

The dispersion relation $\alpha^2-\alpha_s^2 = k_1(c-c_s)+k_2(c-c_s)^2+k_3(c-c_s)^3+\dots$ given in §2 has been applied to investigate the stability of some stratified shear flows. It becomes especially simple when the velocity profile $U(y)$ is an odd function of y, and the buoyancy frequency $N(y)$ an even function, and the neutral mode is stationary. Then k_l is real when l is even, and purely imaginary when l is odd. The particular flows that we have considered have velocity and density profiles of this type.

It is found that if $k_1 = 0$ along some neutral curve where $c_s = 0$ in the (α, J)-plane there may be instability in the vicinity of this curve on one or both sides. It depends on k_2 and $k_3 = \mathrm{i}k_{3i}$. There will always be instability on that side of the neutral curve where k_2 and $\alpha^2-\alpha_s^2$ have opposite signs, and here $c_i \sim (-(\alpha^2-\alpha_s^2)/k_2)^{\frac{1}{2}}$ and $c_r = 0$ when $\alpha \to \alpha_s$. If, in addition, k_2 and k_{3i} have opposite signs there will be instability on the other side of the neutral curve too, and here $c_i \sim -k_{3i}(\alpha^2-\alpha_s^2)/2k_2^2$ and $c_r \sim \pm((\alpha^2-\alpha_s^2)/k_2)^{\frac{1}{2}}$ when $\alpha \to \alpha_s$. On this side there are therefore two unstable waves with opposite directions of propagation.

Application of the general theory to the stratified Couette flow yields results in agreement with the numerical results of Huppert (1973). In this simple case k_2 and k_3 are also obtained by expressing the dispersion relation in terms of two confluent hypergeometric functions, and the general theory is confirmed. However, it turns out that k_2 and k_3 are more easily obtained from the general theory.

For the Garcia flow neutral curves and instability regions that have not been known previously are found. The neutral modes corresponding to these neutral curves are not stationary.

Application of the general theory to the stratified shear flow in §3.2 also reveals instability on both sides of the neutral curves along which $c_s = 0$ and $k_1 = 0$. By numerical calculations neutral curves corresponding to non-stationary neutral modes are obtained.

Both for the Garcia flow and the flow in §3.2 the stability boundary close to the origin in the (α, J)-plane is given by the curve along which $c_s = 0$. This is in agreement with the results of Drazin & Howard (1961, 1963) concerning the stability characteristics of unbounded flows for long waves (cf. Drazin & Howard 1966).

One of the referees has pointed out to us that the theory might be applicable to the Charney model of baroclinic instability (Charney 1947; Pedlosky 1979), where instability occurs on either side of the neutral modes. It is certainly true that the dispersion relation (2.3) can be used to study the stability characteristics in the vicinity of the neutral modes, but the governing equation and the boundary conditions in the Charney model are different from (2.1) and (2.2), and the expressions for k_l, $l = 1, 2, \ldots$, will be modified accordingly. However, in the Charney model the neutral modes are stationary ($c_s = 0$), and k_1 is obviously equal to zero, so an analogous problem to the one we have studied in our paper arises in this model, and it can be solved in the same way.

We are grateful to a referee for drawing our attention to Charney's model.

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